

# Construction of minimal non-abelian left group codes

Gabriela Olteanu and Inneke Van Gelder

## Abstract

Algorithms to construct minimal left group codes are provided. These are based on results describing a complete set of orthogonal primitive idempotents in each Wedderburn component of a semisimple finite group algebra  $\mathbb{F}G$  for a large class of groups  $G$ .

As an illustration of our methods, alternative constructions to some best linear codes over  $\mathbb{F}_2$  and  $\mathbb{F}_3$  are given.

## Index Terms

group codes, linear codes, primitive idempotents, Wedderburn decomposition, finite group algebras

## I. INTRODUCTION

In this paper  $\mathbb{F} = \mathbb{F}_s$  denotes a finite field with  $s$  elements. A linear code over  $\mathbb{F}$  of length  $n$  and rank  $k$  is a linear subspace  $C$  with dimension  $k$  of the vector space  $\mathbb{F}^n$ . The standard basis of  $\mathbb{F}^n$  is denoted by  $E = \{e_1, \dots, e_n\}$ . The vectors in  $C$  are called codewords, the size of a code is the number of codewords and equals  $s^k$ . The distance of a code is the minimum distance between distinct codewords, i.e. the number of elements in which they differ. A linear code of length  $n$ , dimension  $k$ , and distance  $d$  is called a  $[n, k, d]$ -code. Bounds on the minimum distance of linear codes are known, see [Bro98], [Gra06], [Gra07], [BBC<sup>+</sup>12]. A linear code  $C$  can be represented as the  $\mathbb{F}^n$ -span of a minimal set of codewords, these basis codewords are often collated in the rows of a matrix known as a generating matrix for the code  $C$ .

For any group  $G$ , we denote by  $\mathbb{F}G$  the group algebra over  $G$  with coefficients in  $\mathbb{F}$ . If  $G$  is a group of order  $n$  and  $C \subseteq \mathbb{F}^n$  is a linear code, then we say that  $C$  is a left  $G$ -code (respectively a  $G$ -code) if there is a bijection  $\phi : E \rightarrow G$  such that the linear extension of  $\phi$  to an isomorphism  $\phi : \mathbb{F}^n \rightarrow \mathbb{F}G$  maps  $C$  to a left ideal (respectively a two-sided ideal) of  $\mathbb{F}G$ . A left group code (respectively a group code) is a linear code which is a left  $G$ -code (respectively a  $G$ -code) for some group  $G$ . A (left) cyclic group code (respectively, abelian, metacyclic, nilpotent group code, ...) is a linear code which is a (left)  $G$ -code for some cyclic group (respectively, abelian, metacyclic, nilpotent group, ...)  $G$ .

In [SL95], [BdRS09] it is proved that if  $C$  is a metacyclic group code then  $C$  is an abelian group code. Also no example of a group code which is not an abelian group code is known. However, Bernal, del Río and Simón proved that for every non-abelian group  $G$  and every prime  $p$  not dividing the order of  $G$  there is a left  $G$ -code over some field of characteristic  $p$  which is not an abelian group code [BdRS09, Proposition 3.3]. Therefore, we will study (minimal) left group codes. For a metacyclic group

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The research is supported by the grant PN-II-RU-TE-2009-1 project ID\_303, the grant PN-II-RU-TE-2012-3-0109 and by the Research Foundation Flanders (FWO - Vlaanderen).

$G = \langle a, b \mid a^m = 1 = b^n, ba = a^r b \rangle$  where  $\gcd(m, r) = 1$ ,  $r^n \equiv 1 \pmod{m}$  and both  $m$  and  $n$  are odd, Sabin and Lomonaco [SL95] gave an algorithm to determine minimal left codes in  $\mathbb{F}G$  where  $\mathbb{F}$  has characteristic 2. They discovered several good metacyclic codes and they expressed the hope (also inspired by results on other non-abelian codes [CS89]) that more “good” and perhaps even “best” codes may be discovered among the ideals of non-abelian group rings. We will improve such an algorithm for a larger class of groups and fields and discover some best codes.

## II. PRELIMINARIES

When  $R$  is a semisimple ring (i.e.  $R$  is a direct sum of a finite number of minimal left ideals), then every left ideal  $L$  of  $R$  is of the form  $L = Re$ , where  $e$  is an idempotent of  $R$ . Therefore, we can use the idempotents to characterize the decompositions of semisimple rings as a direct sum of minimal left ideals. In particular, let  $R = \bigoplus_{i=1}^t L_i$  be a decomposition of a semisimple ring as a direct sum of minimal left ideals. Then, there exists a family  $\{e_1, \dots, e_t\}$  of elements of  $R$  such that: each  $e_i \neq 0$  is an idempotent element, if  $i \neq j$ , then  $e_i e_j = 0$ ,  $1 = e_1 + \dots + e_t$  and each  $e_i$  cannot be written as  $e_i = e'_i + e''_i$ , where  $e'_i, e''_i$  are idempotents such that  $e'_i, e''_i \neq 0$  and  $e'_i e''_i = 0$ ,  $1 \leq i \leq t$ . Conversely, if there exists a family of idempotents  $\{e_1, \dots, e_t\}$  satisfying the previous conditions, then the left ideals  $L_i = Re_i$  are minimal and  $R = \bigoplus_{i=1}^t L_i$ . Such a set of idempotents is called a complete set of orthogonal primitive idempotents of the ring  $R$ . Note that such a set is not uniquely determined. When studying left group codes, it is useful to study minimal left group codes, i.e. codes associated with minimal left ideals, and in particular primitive idempotents of finite group algebras.

Recall that, given a decomposition of a semisimple ring  $R$  as a direct sum of minimal left ideals, we can group isomorphic left ideals together. The sum of all left ideals isomorphic to one in the decomposition, turns out to be a minimal two-sided ideal of  $R$  which is simple as a ring. Also the decomposition of  $R$  as a direct sum of two-sided ideals is related to a family of idempotents. Let  $R = \bigoplus_{i=1}^s A_i$  be a decomposition of a semisimple ring as a direct sum of minimal two-sided ideals. Then, there exists a family  $\{e_1, \dots, e_s\}$  of elements of  $R$  such that: each  $e_i \neq 0$  is a central idempotent element, if  $i \neq j$ , then  $e_i e_j = 0$ ,  $1 = e_1 + \dots + e_s$  and each  $e_i$  cannot be written as  $e_i = e'_i + e''_i$ , where  $e'_i, e''_i$  are central idempotents such that  $e'_i, e''_i \neq 0$  and  $e'_i e''_i = 0$ ,  $1 \leq i \leq s$ . The elements  $\{e_1, \dots, e_s\}$  are called the primitive central idempotents of  $R$  and they give rise to the well-known Wedderburn-Artin Theorem. Using this knowledge backwards, it can be helpful to consider the Wedderburn decomposition and the primitive central idempotents in order to determine a complete set of orthogonal primitive idempotents.

From now on,  $G$  denotes an arbitrary finite group such that  $\mathbb{F}G$  is semisimple. By Maschke’s Theorem this is equivalent to saying that the order of  $G$  is coprime to the characteristic of  $\mathbb{F}$ . The notation  $H \leq G$  (resp.  $H \trianglelefteq G$ ) means that  $H$  is a subgroup (resp. normal subgroup) of  $G$ . For  $H \leq G$ ,  $g \in G$  and  $h \in H$ , we define  $H^g = g^{-1}Hg$  and  $h^g = g^{-1}hg$ . Analogously, for  $\alpha \in \mathbb{F}G$  and  $g \in G$ ,  $\alpha^g = g^{-1}\alpha g$ . For  $H \leq G$ ,  $N_G(H)$  denotes the normalizer of  $H$  in  $G$  and we set  $\tilde{H} = |H|^{-1} \sum_{h \in H} h$ , an idempotent of  $\mathbb{F}G$ , and if  $H = \langle g \rangle$  then we simply write  $\tilde{g}$  for  $\tilde{\langle g \rangle}$ .

The classical method for computing primitive central idempotents in a semisimple group algebra  $\mathbb{F}G$  involves characters of the group  $G$ . All the characters of any finite group are assumed to be characters in  $\overline{\mathbb{F}}$ , a fixed algebraic closure of the field  $\mathbb{F}$ . For an irreducible character  $\chi$  of  $G$ ,  $e(\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$  is the primitive central idempotent of  $\overline{\mathbb{F}}G$  associated to  $\chi$  and  $e_{\overline{\mathbb{F}}}(\chi)$  is the only primitive central idempotent  $e$  of  $\mathbb{F}G$  such that  $\chi(e) \neq 0$ . The field of character values of  $\chi$  over  $\mathbb{F}$  is defined as  $\mathbb{F}(\chi) = \mathbb{F}(\chi(g) : g \in G)$ , that is the field extension of  $\mathbb{F}$  generated over  $\mathbb{F}$  by the image of  $\chi$ . The automorphism group  $\text{Aut}(\overline{\mathbb{F}})$  acts on  $\overline{\mathbb{F}}G$  by acting on the coefficients, that is  $\sigma \sum_{g \in G} a_g g = \sum_{g \in G} \sigma(a_g)g$ , for  $\sigma \in \text{Aut}(\overline{\mathbb{F}})$  and  $a_g \in \overline{\mathbb{F}}$ . Following [Yam73], we know that  $e_{\mathbb{F}}(\chi) = \sum_{\sigma \in \text{Gal}(\mathbb{F}(\chi)/\mathbb{F})} \sigma e(\chi)$ .

New methods for the computation of the primitive central idempotents in a group algebra do not involve characters. The main ingredient in this theory is the following element, introduced in [OdRS04]. If  $K \trianglelefteq H \leq G$ , then let  $\varepsilon(H, K)$  be the element of  $\mathbb{Q}H \subseteq \mathbb{Q}G$  defined as

$$\varepsilon(H, K) = \begin{cases} \widetilde{K} & \text{if } H = K, \\ \prod_{M/K \in \mathcal{M}(H/K)} (\widetilde{K} - \widetilde{M}) & \text{if } H \neq K, \end{cases}$$

where  $\mathcal{M}(H/K)$  denotes the set of minimal normal non-trivial subgroups of  $H/K$ . Furthermore,  $e(G, H, K)$  denotes the sum of the different  $G$ -conjugates of  $\varepsilon(H, K)$ . By [OdRS04, Theorem 4.4], the elements  $\varepsilon(H, K)$  are the building blocks for the primitive central idempotents of  $\mathbb{Q}G$  for abelian-by-supersolvable groups  $G$ .

We introduce some notations and results from [BdR07]. Let  $\mathbb{F}$  and  $G$  be as before, with  $|G| = n$ . Throughout the paper, we fix an algebraic closure of  $\mathbb{F}$ , denoted by  $\overline{\mathbb{F}}$ . For every positive integer  $k$  coprime with  $s$ ,  $\xi_k$  denotes a primitive  $k$ -th root of unity in  $\overline{\mathbb{F}}$  and  $o_k(s)$  denotes the multiplicative order of  $s$  modulo  $k$ . Recall that  $\mathbb{F}(\xi_k) \simeq \mathbb{F}_{s^{o_k(s)}}$ , the field of order  $s^{o_k(s)}$ . Let  $\mathcal{Q}$  denote the subgroup of  $\mathbb{Z}_n^*$ , the group of units of the ring  $\mathbb{Z}_n$ , generated by the class of  $s$  and consider  $\mathcal{Q}$  acting on  $G$  by  $t \cdot g = g^t$ . The  $s$ -cyclotomic classes of  $G$  are the orbits of  $G$  under the action of  $\mathcal{Q}$  on  $G$ . Let  $G^*$  be the group of irreducible characters in  $\overline{\mathbb{F}}$  of  $G$ . Now let  $\mathcal{C}(G)$  denote the set of  $s$ -cyclotomic classes of  $G^*$ , which consist of linear faithful characters of  $G$ .

Let  $K \trianglelefteq H \leq G$  be such that  $H/K$  is cyclic of order  $k$  and  $C \in \mathcal{C}(H/K)$ . If  $\chi \in C$  and  $\text{tr} = \text{tr}_{\mathbb{F}(\xi_k)/\mathbb{F}}$  denotes the field trace of the Galois extension  $\mathbb{F}(\xi_k)/\mathbb{F}$ , then we set

$$\varepsilon_C(H, K) = |H|^{-1} \sum_{h \in H} \text{tr}(\chi(hK)) h^{-1} = [H : K]^{-1} \widetilde{K} \sum_{X \in H/K} \text{tr}(\chi(X)) h_X^{-1},$$

where  $h_X$  denotes a representative of  $X \in H/K$ . Note that  $\varepsilon_C(H, K)$  does not depend on the choice of  $\chi \in C$ . Furthermore,  $e_C(G, H, K)$  denotes the sum of the different  $G$ -conjugates of  $\varepsilon_C(H, K)$ . Note that the elements  $\varepsilon_C(H, K)$  will occur in Theorem II.2 as the building blocks for the primitive central idempotents of finite group algebras.

If  $H$  is a subgroup of  $G$ ,  $\psi$  a linear character of  $H$  and  $g \in G$ , then  $\psi^g$  denotes the character of  $H^g$  given by  $\psi^g(h^g) = \psi(h)$ . This defines an action of  $G$  on the set of linear characters of subgroups of  $G$ . Note that if  $K = \text{Ker } \psi$ , then  $\text{Ker } \psi^g = K^g$  and therefore the rule  $\psi \mapsto \psi^g$  defines a bijection between the set of linear characters of  $H$  with kernel  $K$  and the set of linear characters of  $H^g$  with kernel  $K^g$ . This bijection maps  $s$ -cyclotomic classes to  $s$ -cyclotomic classes and hence induces a bijection  $\mathcal{C}(H/K) \rightarrow \mathcal{C}(H^g/K^g)$ .

Let  $K \trianglelefteq H \leq G$  be such that  $H/K$  is cyclic. Then the action from the previous paragraph induces an action of  $N = N_G(H) \cap N_G(K)$  on  $\mathcal{C}(H/K)$  and it is easy to see that the stabilizer of a cyclotomic class in  $\mathcal{C}(H/K)$  is independent of the cyclotomic class. We denote by  $E_G(H/K)$  the stabilizer of such (and thus of any) cyclotomic class in  $\mathcal{C}(H/K)$  under this action.

**Remark II.1.** The set  $E_G(H/K)$  can be determined without the need to use characters. Let  $K \trianglelefteq H \leq G$  be such that  $H/K$  is cyclic. Then  $N = N_G(H) \cap N_G(K)$  acts on  $H/K$  by conjugation and this induces an action of  $N$  on the set of  $s$ -cyclotomic classes of  $H/K$ . It is easy to verify that the stabilizers of all the  $s$ -cyclotomic classes of  $H/K$  containing generators of  $H/K$  are equal and coincide with  $E_G(H/K)$ .

A strong Shoda pair of  $G$  is a pair  $(H, K)$  of subgroups of  $G$  satisfying the following conditions:

- (SS1)  $K \leq H \trianglelefteq N_G(K)$ ,
- (SS2)  $H/K$  is cyclic and a maximal abelian subgroup of  $N_G(K)/H$ , and
- (SS3) for every  $g \in G \setminus N_G(K)$ ,  $\varepsilon(H, K)\varepsilon(H, K)^g = 0$ .

It is also proven in [BdR07] that  $\text{Cen}_G(\varepsilon_C(H, K)) = E_G(H/K)$  in the case when  $(H, K)$  is a strong Shoda pair of  $G$ . The following Theorem gives a description of some primitive central idempotents of  $\mathbb{F}G$  and the associated simple components.

**Theorem II.2.** [BdR07, Theorem 7] *Let  $G$  be a finite group and  $\mathbb{F}$  a finite field of order  $s$  such that  $\mathbb{F}G$  is semisimple. Let  $(H, K)$  be a strong Shoda pair of  $G$  and  $C \in \mathcal{C}(H/K)$ . Then  $e_C(G, H, K)$  is a primitive central idempotent of  $\mathbb{F}G$  and*

$$\mathbb{F}Ge_C(G, H, K) \simeq M_{[G:H]}(\mathbb{F}_{q^o/[E:H]}),$$

where  $E = E_G(H/K)$  and  $o$  is the multiplicative order of  $s$  modulo  $[H : K]$ .

**Remark II.3.** From [BdR07, Theorem 7], we also know that there is a strong relation between the primitive central idempotents in a rational group algebra  $\mathbb{Q}G$  and the primitive central idempotents in a finite group algebra  $\mathbb{F}G$  that makes use of the strong Shoda pairs of  $G$ . More precisely, if  $X$  is a set of strong Shoda pairs of  $G$  and every primitive central idempotent of  $\mathbb{Q}G$  is of the form  $e(G, H, K)$  for  $(H, K) \in X$ , then every primitive central idempotent of  $\mathbb{F}G$  is of the form  $e_C(G, H, K)$  for  $(H, K) \in X$  and  $C \in \mathcal{C}(H/K)$ .

Let  $\chi$  be an irreducible (complex) character of  $G$ . Then  $\chi$  is strongly monomial if there is a strong Shoda pair  $(H, K)$  of  $G$  and a linear character  $\theta$  of  $H$  with kernel  $K$  such that  $\chi = \theta^G$ , the induced character of  $G$ . The group  $G$  is strongly monomial if every irreducible character of  $G$  is strongly monomial.

A complete description of the primitive central idempotents and the simple components for strongly monomial groups is given in [BdR07].

**Corollary II.4.** *If  $G$  is a strongly monomial group and  $\mathbb{F}$  is a finite field of order  $s$  such that  $\mathbb{F}G$  is semisimple, then every primitive central idempotent of  $\mathbb{F}G$  is of the form  $e_C(G, H, K)$  for  $(H, K)$  a strong Shoda pair of  $G$  and  $C \in \mathcal{C}(H/K)$ . Furthermore, for every strong Shoda pair  $(H, K)$  of  $G$  and every  $C \in \mathcal{C}(H/K)$ ,  $\mathbb{F}Ge_C(G, H, K) \simeq M_{[G:H]}(\mathbb{F}_{s^o/[E:H]})$ , where  $E = E_G(H/K)$  and  $o$  is the multiplicative order of  $s$  modulo  $[H : K]$ .*

However, in some cases we have more information on the algebra isomorphism given in the previous Theorem. We can express the simple algebra  $\mathbb{F}Ge_C(G, H, K)$  in terms of a crossed product.

If  $R$  is a (not necessarily finite) unital associative ring and  $G$  is a group then  $R *_{\tau}^{\alpha} G$  denotes a crossed product with action  $\alpha : G \rightarrow \text{Aut}(R)$  and twisting (a two-cocycle)  $\tau : G \times G \rightarrow \mathcal{U}(R)$  (see for example [Pas89]), i.e.  $R *_{\tau}^{\alpha} G$  is the associative ring  $\bigoplus_{g \in G} Ru_g$  with multiplication given by the following rules:  $u_g a = \alpha_g(a) u_g$  and  $u_g u_h = \tau(g, h) u_{gh}$ , for  $a \in R$  and  $g, h \in G$ . Recall that a classical crossed product is a crossed product  $L *_{\tau}^{\alpha} G$ , where  $L/F$  is a finite Galois extension (of not necessarily finite fields),  $G = \text{Gal}(L/F)$  is the Galois group of the field extension  $L/F$  and  $\alpha$  is the natural action of  $G$  on  $L$ . A classical crossed product  $L *_{\tau}^{\alpha} G$  is denoted by  $(L/F, \tau)$  [Rei75]. If the twisting  $\tau$  is cohomologically trivial, then the classical crossed product is isomorphic to a matrix algebra over its center. Moreover, when  $\tau = 1$  we get an explicit isomorphism. More precisely, denoting the matrix associated to an endomorphism  $f$  in a basis  $B$  as  $[f]_B$ , we have the following result.

**Theorem II.5.** [Rei75, Corollary 29.8] *Let  $L/F$  be a finite Galois extension and  $n = [L : F]$ . The classical crossed product  $(L/F, 1)$  is isomorphic (as  $F$ -algebra) to  $M_n(F)$ . Moreover, an isomorphism is given by*

$$\begin{aligned} \psi : (L/F, 1) &\longrightarrow \text{End}_F(L) &\longrightarrow M_n(F) \\ xu_{\sigma} &\longmapsto x' \circ \sigma &\longmapsto [x' \circ \sigma]_B, \end{aligned}$$

for  $x \in L$ ,  $\sigma \in \text{Gal}(L/F)$ ,  $B$  an  $F$ -basis of  $L$  and where  $x'$  denotes multiplication by  $x$  on  $L$ .

Let  $(H, K)$  be a strong Shoda pair of a group  $G$ ,  $C \in \mathcal{C}(H/K)$ ,  $E = E_G(H/K)$  and  $\phi : E/H \rightarrow E/K$  a left inverse of the canonical projection  $E/K \rightarrow E/H$ . As mentioned in the proofs of [BdR07], with ideas from [OdRS04], we know that  $\mathbb{F}E\varepsilon_C(H, K) = \mathbb{F}H\varepsilon_C(H, K) *_{\tau}^{\alpha} E/H = \mathbb{F}(\zeta_{[H:K]}) *_{\tau}^{\alpha} E/H$  and the action and twisting are given by

$$\begin{aligned}\alpha_{gH}(\zeta_{[H:K]}) &= \zeta_{[H:K]}^i, \text{ if } yK^{\phi(gH)} = y^iK \text{ and} \\ \tau(gH, g'H) &= \zeta_{[H:K]}^j, \text{ if } \phi(gg'H)^{-1}\phi(gH)\phi(g'H) = y^jK,\end{aligned}$$

for  $gH, g'H \in E/H$  and integers  $i$  and  $j$ . Since the action  $\alpha$  is faithful,  $\mathbb{F}(\zeta_{[H:K]}) *_{\tau}^{\alpha} E/H$  can be described as a classical crossed product  $(\mathbb{F}(\zeta_{[H:K]})/Z, \tau)$ , where  $Z = \mathbb{F}_{s^o/[E:H]}$  is the center of the algebra, which is determined by the Galois action  $\alpha$ . Hence  $E/H \simeq \text{Gal}(\mathbb{F}(\zeta_{[H:K]})/Z)$ . If moreover the twisting  $\tau$  is trivial, we know a concrete isomorphism  $\mathbb{F}Ge_C(G, H, K) \simeq M_{[G:E]}(\mathbb{F}E\varepsilon_C(H, K)) \simeq M_{[G:H]}(\mathbb{F}_{s^o/[E:H]}).$

Using the description of the primitive central idempotents and the Wedderburn components of a semisimple finite group algebra  $\mathbb{F}G$ , we were able to describe a complete set of orthogonal primitive idempotents of  $\mathbb{F}G$  in the case where  $G$  is nilpotent [OVG11]. This description will be used in section V to construct minimal left nilpotent group codes.

**Theorem II.6.** [OVG11, Theorem 3.3] *Let  $\mathbb{F}$  be a finite field and  $G$  a finite nilpotent group such that  $\mathbb{F}G$  is semisimple. Let  $(H, K)$  be a strong Shoda pair of  $G$ ,  $C \in \mathcal{C}(H/K)$  and set  $e_C = e_C(G, H, K)$ ,  $\varepsilon_C = \varepsilon_C(H, K)$ ,  $H/K = \langle \bar{a} \rangle$ ,  $E = E_G(H/K)$ . Let  $E_2/K$  and  $H_2/K = \langle \bar{a}_2 \rangle$  (respectively  $E_{2'}/K$  and  $H_{2'}/K = \langle \bar{a}_{2'} \rangle$ ) denote the 2-parts (respectively 2'-parts) of  $E/K$  and  $H/K$  respectively. Then  $\langle \bar{a}_{2'} \rangle$  has a cyclic complement  $\langle \bar{b}_{2'} \rangle$  in  $E_{2'}/K$ .*

*A complete set of orthogonal primitive idempotents of  $\mathbb{F}Ge_C$  consists of the conjugates of  $\beta_{e_C} = \widetilde{b_{2'}}\beta_{2\varepsilon_C}$  by the elements of  $T_{e_C} = T_{2'}T_2T_E$ , where  $T_{2'} = \{1, a_{2'}, a_{2'}^2, \dots, a_{2'}^{[E_{2'}:H_{2'}]-1}\}$ ,  $T_E$  denotes a right transversal of  $E$  in  $G$  and  $\beta_2$  and  $T_2$  are given according to the cases below.*

- 1) *If  $H_2/K$  has a complement  $M_2/K$  in  $E_2/K$  then  $\beta_2 = \widetilde{M_2}$ . Moreover, if  $M_2/K$  is cyclic, then there exists  $b_2 \in E_2$  such that  $E_2/K$  is given by the following presentation*

$$\langle \bar{a}_2, \bar{b}_2 \mid \bar{a}_2^{2^n} = \bar{b}_2^{2^k} = 1, \bar{a}_2\bar{b}_2 = \bar{a}_2^r \rangle,$$

*and if  $M_2/K$  is not cyclic, then there exist  $b_2, c_2 \in E_2$  such that  $E_2/K$  is given by the following presentation*

$$\langle \bar{a}_2, \bar{b}_2, \bar{c}_2 \mid \bar{a}_2^{2^n} = \bar{b}_2^{2^k} = \bar{c}_2^2 = 1, \bar{a}_2\bar{b}_2 = \bar{a}_2^r, \bar{a}_2\bar{c}_2 = \bar{a}_2^{-1}, [\bar{b}_2, \bar{c}_2] = 1 \rangle,$$

*with  $r \equiv 1 \pmod{4}$  (or equivalently  $\bar{a}_2^{2^{n-2}}$  is central in  $E_2/K$ ). Then*

- a)  $T_2 = \{1, a_2, a_2^2, \dots, a_2^{2^k-1}\}$ , *if  $\bar{a}_2^{2^{n-2}}$  is central in  $E_2/K$  (unless  $n \leq 1$ ) and  $M_2/K$  is cyclic; and*
  - b)  $T_2 = \{1, a_2, a_2^2, \dots, a_2^{d/2-1}, a_2^{2^{n-2}}, a_2^{2^{n-2}+1}, \dots, a_2^{2^{n-2}+d/2-1}\}$ , *where  $d = [E_2 : H_2]$ , otherwise.*
- 2) *If  $H_2/K$  has no complement in  $E_2/K$ , then there exist  $b_2, c_2 \in E_2$  such that  $E_2/K$  is given by the following presentation*

$$\begin{aligned}\langle \bar{a}_2, \bar{b}_2, \bar{c}_2 \mid & \bar{a}_2^{2^n} = \bar{b}_2^{2^k} = 1, \bar{c}_2^2 = \bar{a}_2^{2^{n-1}}, \bar{a}_2\bar{b}_2 = \bar{a}_2^r, \\ & \bar{a}_2\bar{c}_2 = \bar{a}_2^{-1}, [\bar{b}_2, \bar{c}_2] = 1 \rangle,\end{aligned}$$

*with  $r \equiv 1 \pmod{4}$ . In this case,  $\beta_2 = \widetilde{b_2 \frac{1+xa_2^{2^{n-2}}+ya_2^{2^{n-2}}c_2}{2}}$  and*

$$T_2 = \{1, a_2, a_2^2, \dots, a_2^{2^k-1}, c_2, c_2a_2, c_2a_2^2, \dots, c_2a_2^{2^k-1}\},$$

with  $x, y \in \mathbb{F}$ , satisfying  $x^2 + y^2 = -1$  and  $y \neq 0$ .

This theorem provided a straightforward implementation in GAP. Nevertheless, in case (2), there might occur some difficulties finding solutions for this equation. However, we were able to overcome this problem ([OVG11, Remark 3.4]). Computations involving strong Shoda pairs and primitive central idempotents were already provided in the GAP package Wedderga [BKO<sup>+</sup>13] and we've included our new algorithms there.

### III. A COMPLETE SET OF ORTHOGONAL PRIMITIVE IDEMPOTENTS IN $\mathbb{F}G$

Throughout this section we will assume that  $\mathbb{F}$  is a finite field of order  $s$  and  $G$  is a finite group such that the order of  $G$  is coprime to  $s$ . We will focus on simple components of  $\mathbb{F}G$  which are determined by a strong Shoda pair  $(H, K)$  and a class  $C \in \mathcal{C}(H/K)$  such that  $\tau(gH, g'H) = 1$  for all  $g, g' \in E = E_G(H/K)$  (with notation as in Proposition II.2). For such a component, we describe a complete set of orthogonal primitive idempotents. This construction is based on the isomorphism of Theorem II.5 on classical crossed products with trivial twisting. Such a description, together with the description of the primitive central idempotent  $e_C = e_C(G, H, K)$  determining the simple component, yields a complete set of irreducible modules and will be applied in section V to construct codes.

Before we do so, we need a basis of  $\mathbb{F}(\zeta_{[H:K]})/\mathbb{F}(\zeta_{[H:K]})^{E/H} = \mathbb{F}_{s^o}/\mathbb{F}_{s^o/[E:H]}$  (with  $o$  the multiplicative order of  $s$  modulo  $[H:K]$ ) of the form  $\{w^x \mid x \in E/H\}$  with  $w \in \mathbb{F}(\zeta_{[H:K]})$ . That such a basis exists follows from the well-known Normal Basis Theorem which states that if  $K/F$  is a finite Galois extension, then there exists an element  $w \in K$  such that  $\{\sigma(w) \mid \sigma \in \text{Gal}(K/F)\}$  is an  $F$ -basis of  $K$ , a so-called normal basis, whence  $w$  is called normal in  $K/F$ . Recall that  $E/H$ , the Galois group of  $\mathbb{F}_{s^o}$  over  $\mathbb{F}_{s^o/[E:H]}$ , is cyclic and generated by the Frobenius automorphism  $x \mapsto x^{s^o/[E:H]}$  (see [Rom06]). Hence if  $\beta \in \mathbb{F}_{s^o}$  is such that the  $[E:H]$  elements  $\{\beta, \beta^{s^o/[E:H]}, \dots, \beta^{(s^o/[E:H])^{[E:H]-1}}\}$  are linearly independent, then this set forms a normal basis for  $\mathbb{F}_{s^o}$  over  $\mathbb{F}_{s^o/[E:H]}$ . For a background on the construction of normal bases, see Artin [Art73], Lüneburg [Lün86], Lenstra [Len91] and Gao [Gao93]. The construction of normal bases is implemented in GAP in the method `NormalBase`.

Now we can state our main result on primitive idempotents.

**Theorem III.1.** *Let  $G$  be a finite group and  $\mathbb{F}$  a finite field of order  $s$  such that  $s$  is coprime to the order of  $G$ . Let  $(H, K)$  be a strong Shoda pair of  $G$  such that  $\tau(gH, g'H) = 1$  for all  $g, g' \in E = E_G(H/K)$ , and let  $C \in \mathcal{C}(H/K)$ . Let  $\varepsilon = \varepsilon_C(H, K)$  and  $e = e_C(G, H, K)$ . Let  $w$  be a normal element of  $\mathbb{F}_{s^o}/\mathbb{F}_{s^o/[E:H]}$  (with  $o$  the multiplicative order of  $s$  modulo  $[H:K]$ ) and  $B$  the normal basis determined by  $w$ . Let  $\psi$  be the isomorphism between  $\mathbb{F}E\varepsilon$  and the matrix algebra  $M_{[E:H]}(\mathbb{F}_{s^o/[E:H]})$  with respect to the basis  $B$  as stated in Theorem II.5. Let  $P, A \in M_{[E:H]}(\mathbb{F}_{s^o/[E:H]})$  be the matrices*

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then

$$\{\widehat{xT_1\varepsilon x^{-1}} \mid x \in T_2 \langle x_e \rangle\}$$



is a complete set of orthogonal primitive idempotents of  $\mathbb{F}Ge$  where  $x_e = \psi^{-1}(PAP^{-1})$ ,  $T_1$  is a transversal of  $H$  in  $E$  and  $T_2$  is a right transversal of  $E$  in  $G$ . By  $\widehat{T_1}$  we denote the element  $\frac{1}{|T_1|} \sum_{t \in T_1} t$  in  $\mathbb{F}G$ .

*Proof.* Consider the simple component

$$\mathbb{F}Ge \simeq M_{[G:E]}(\mathbb{F}E\varepsilon) \simeq M_{[G:H]}(\mathbb{F}_{so/[E:H]})$$

of  $\mathbb{F}G$ . Without loss of generality we may assume that  $G = E$ . Indeed, if we obtain a complete set of orthogonal primitive idempotents of  $\mathbb{F}E\varepsilon$ , then the conjugates by the transversal  $T_2$  of  $E$  in  $G$  will give a complete set of orthogonal primitive idempotents of  $\mathbb{F}Ge$  since  $e = \sum_{t \in T_2} \varepsilon^t$  and different  $\varepsilon^t$ 's are orthogonal.

From now on we assume that  $G = E$  and  $e = \varepsilon$  and denote  $n = [E : H]$ . Then  $B = \{w^{gH} : g \in T_1\}$ . Since  $G/H$  acts on  $\mathbb{F}He$  via the induced conjugation action on  $H/K$ , it is easily seen that the action of  $G/H$  on  $B$  is regular. Hence it is readily verified that for each  $g \in T_1$ ,  $\psi(g\varepsilon)$  is a permutation matrix, and

$$\psi(\widehat{T_1}e) = \frac{1}{n} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Clearly  $\psi(\widehat{T_1}e)$  has eigenvalues 1 and 0, with respective eigenspaces  $V_1 = \text{vect}\{(1, 1, \dots, 1)\}$  and  $V_0 = \text{vect}\{(1, -1, 0, \dots, 0), (1, 0, -1, \dots, 0), \dots, (1, 0, 0, \dots, -1)\}$ , where  $\text{vect}(S)$  denotes the vector space generated by the set  $S$ . Hence

$$\psi(\widehat{T_1}e) = PE_{11}P^{-1},$$

where we denote by  $E_{ij} \in M_n(\mathbb{F}_{so/[E:H]})$  the matrices whose entries are all 0 except in the  $(i, j)$ -spot, where it is 1. One knows that  $\{E_{11}, E_{22}, \dots, E_{nn}\}$  and hence also

$$\{\psi(\widehat{T_1}e) = PE_{11}P^{-1}, PE_{22}P^{-1}, \dots, PE_{nn}P^{-1}\}$$

forms a complete set of orthogonal primitive idempotents of  $M_n(\mathbb{F}_{so/[E:H]})$ . Let  $y = \psi(x_e) = PAP^{-1}$ . As

$$E_{22} = AE_{11}A^{-1}, \dots, E_{nn} = A^{n-1}E_{11}A^{-n+1}$$

we obtain that

$$\{\psi(\widehat{T_1}e), y\psi(\widehat{T_1}e)y^{-1}, \dots, y^{n-1}\psi(\widehat{T_1}e)y^{-n+1}\}$$

forms a complete set of orthogonal primitive idempotents of  $M_n(\mathbb{F}_{so/[E:H]})$ . Hence, applying  $\psi^{-1}$  gives us a complete set of orthogonal primitive idempotents of  $\mathbb{F}Ge$ .  $\square$

This method yields a detailed description of a complete set of orthogonal primitive idempotents of  $\mathbb{F}G$  when  $G$  is a strongly monomial group such that there exists a complete and non-redundant set of strong Shoda pairs  $(H, K)$  satisfying  $\tau(gH, g'H) = 1$  for all  $g, g' \in E_G(H/K)$ . Remark that similar techniques are used in [JdROVG] to construct a complete set of orthogonal primitive idempotents of the rational group algebra  $\mathbb{Q}G$  with  $G$  as before. For example, the symmetric group  $S_4$  and the alternating group  $A_4$  of degree 4 have a trivial twisting in all Wedderburn components of their group rings. Trivially, all abelian groups are included and it is also easy to prove that for all dihedral groups  $D_{2n} = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1} \rangle$

there exists a complete and non-redundant set of strong Shoda pairs with trivial twisting since the group action involved has order 2 and hence is faithful. On the other hand, for quaternion groups  $Q_{4n} = \langle x, y \mid x^{2n} = y^4 = 1, x^n = y^2, x^y = x^{-1} \rangle$ , one can verify that the strong Shoda pair  $(\langle x \rangle, 1)$  yields a non-trivial twisting.

Even when the group is not strongly monomial or some strong Shoda pairs yield a non-trivial twisting, our description of primitive idempotents can still be used in the components determined by a strong Shoda pair with trivial twisting. This implies that we can always compute some minimal left  $G$ -codes over a finite field  $\mathbb{F}$  for a finite group  $G$  of order coprime to  $\text{char}(\mathbb{F})$ .

#### IV. A CLASS OF METACYCLIC GROUPS

In this section, we show that our main result can be applied to the metacyclic groups of the form  $C_{q^m} \rtimes C_{p^n}$  with  $C_{p^n}$  acting faithfully on  $C_{q^m}$  and  $p$  and  $q$  different primes, and fields  $\mathbb{F}$  of size  $s$  coprime to  $pq$ .

Throughout this section  $p$  and  $q$  are different primes,  $m$  and  $n$  are positive integers and  $G = \langle a \rangle \rtimes \langle b \rangle$  with  $|a| = q^m$ ,  $|b| = p^n$  and  $\langle b \rangle$  acts faithfully on  $\langle a \rangle$  (i.e.  $C_{\langle b \rangle}(a) = 1$ ). Let  $\sigma$  be the automorphism of  $\langle a \rangle$  given by  $\sigma(a) = a^b$  and assume that  $\sigma(a) = a^r$  with  $r \in \mathbb{Z}$ . As the kernel of the restriction map  $\text{Aut}(\langle a \rangle) \rightarrow \text{Aut}(\langle a^{q^{m-1}} \rangle)$  has order  $q^{m-1}$ , it intersects  $\langle \sigma \rangle$  trivially and therefore the restriction of  $\sigma$  to  $\langle a^{q^{m-1}} \rangle$  also has order  $p^n$ . This implies that  $q \equiv 1 \pmod{p^n}$  and thus  $q$  is odd. Therefore,  $\text{Aut}(\langle a^{q^j} \rangle)$  is cyclic for every  $j = 0, 1, \dots, m$  and  $\langle \sigma \rangle$  is the unique subgroup of  $\text{Aut}(\langle a \rangle)$  of order  $p^n$ . So, for every  $i = 1, \dots, m$ , the image of  $r$  in  $\mathbb{Z}/q^i\mathbb{Z}$  generates the unique subgroup of  $\mathcal{U}(\mathbb{Z}/q^i\mathbb{Z})$  of order  $p^n$ . In particular,  $r^{p^n} \equiv 1 \pmod{q^m}$  and  $r^{p^j} \not\equiv 1 \pmod{q}$  for every  $j = 0, \dots, n-1$ . Therefore,  $r \not\equiv 1 \pmod{q}$  and hence  $G' = \langle a^{r-1} \rangle = \langle a \rangle$ .

In [OdRS04] more information was obtained on the strong Shoda pairs needed to describe the primitive central idempotents of the rational (and hence of a semisimple finite) group algebra of a finite metabelian group. We recall the statement.

**Theorem IV.1.** [OdRS04, Theorem 4.7] *Let  $G$  be a finite metabelian group and let  $A$  be a maximal abelian subgroup of  $G$  containing the commutator subgroup  $G'$ . The primitive central idempotents of  $\mathbb{Q}G$  are the elements of the form  $e(G, H, K)$ , where  $(H, K)$  is a pair of subgroups of  $G$  satisfying the following conditions:*

- 1)  $H$  is a maximal element in the set  $\{B \leq G \mid A \leq B \text{ and } B' \leq K \leq B\}$ ;
- 2)  $H/K$  is cyclic.

Using this description of the strong Shoda pairs, we get a complete and non-redundant set of strong Shoda pairs of  $G$  consisting of two types:

- (i)  $(G, L_i := \langle a, b^{p^i} \rangle)$ ,  $i = 0, \dots, n$ ,
- (ii)  $(\langle a \rangle, K_j := \langle a^{q^j} \rangle)$ ,  $j = 1, \dots, m$ .

It is easy to verify that for these strong Shoda pairs the corresponding twisting is trivial. Hence we can describe a complete set of orthogonal primitive idempotents in each simple component of  $\mathbb{F}G$  using Theorem III.1.

#### V. EXAMPLES OF MINIMAL LEFT GROUP CODES

In this section we will provide some illustrative examples of minimal left group codes making use of the computation of primitive idempotents. For these examples we used the computer algebra system



GAP [GAP13] and the packages GUAVA [BBC<sup>+</sup>12] and Wedderga [BKO<sup>+</sup>13]. The implementation of the used methods (based on Theorem II.6 and Theorem III.1) is now included in Wedderga.

Note that each element  $c$  in  $\mathbb{F}G$  is of the form  $c = \sum_{i=1}^n f_i g_i$ , where we fix an ordering  $\{g_1, g_2, \dots, g_n\}$  of the group elements of  $G$  and  $f_i \in \mathbb{F}$ . If we look at  $c$  as a codeword, we will rather write  $[f_1 f_2 \dots f_n]$ .

**Example V.1.** We define the finite group algebra  $\mathbb{F}_2 G$  over the nilpotent metacyclic group

$$G = \langle a, b \mid a^9 = 1, b^3 = 1, ba = a^4b \rangle$$

and fix an ordering  $S$  of  $G$ .

```
gap> A:=FreeGroup("a", "b");; a:=A.1;; b:=A.2;;
gap> G:=A/[a^9, b^3, b*a*b^(-1)*a^(-4)];;
gap> IsNilpotent(G);
true
gap> F:=GF(2);;
gap> FG:=GroupRing(F, G);;
gap> S:=AsSet(G);
[ <identity ...>, a, b, a^3, a^2, a*b, a^4, b^2, a^3*b, a^6, a^2*b, a^5,
  a*b^2, a^4*b, a^7, a^3*b^2, a^6*b, a^2*b^2, a^5*b, a^8, a^4*b^2, a^7*b,
  a^6*b^2, a^5*b^2, a^8*b, a^7*b^2, a^8*b^2 ]
```

Computing the Wedderburn decomposition of  $\mathbb{F}_2 G$ , we see that only one simple component ( $M_3(\mathbb{F}_4)$ ) will contribute to a non-abelian left group code, given by the strong Shoda pair  $(H, K)$ . To define the primitive central idempotent of  $\mathbb{F}_2 G$  associated to this simple component, we have to define a  $s$ -cyclotomic class of irreducible  $\mathbb{F}_2$ -characters of  $H/K$ , which consists of linear faithful characters. All these linear faithful characters are defined by sending the generator of  $H/K$  to a power (coprime to  $[H : K]$ ) of a fixed primitive  $[H : K]$ -root of unity. Using the generator of  $H/K$ , such a cyclotomic class can be represented by a  $s$ -cyclotomic class modulo  $[H : K]$ , which consists of integers coprime to  $[H : K]$ . With this information we can compute a complete set of orthogonal primitive idempotents in the simple component  $\mathbb{F}_2 Ge_C(G, H, K)$ .

```
gap> WedderburnDecomposition(FG);
[ ( GF(2)^[ 1, 1 ] ), ( GF(2^2)^[ 1, 1 ] ), ( GF(2^2)^[ 1, 1 ] ),
  ( GF(2^2)^[ 1, 1 ] ), ( GF(2^2)^[ 1, 1 ] ), ( GF(2^2)^[ 3, 3 ] ) ]
gap> H:=StrongShodaPairs(G)[6][1];
Group([ b, a^3 ])
gap> K:=StrongShodaPairs(G)[6][2];
Group([ b, a^-2*b^-1*a^-1 ])
gap> N:=Normalizer(G, K);;
gap> epi:=NaturalHomomorphismByNormalSubgroup(N, K);;
gap> QHK:=Image(epi, H);;
gap> gq:=MinimalGeneratingSet(QHK)[1];;
gap> C:=CyclotomicClasses(Size(F), Index(H, K))[2];
[ 1, 2 ]
gap> P:=PrimitiveIdempotentsNilpotent(FG, H, K, C, [epi, gq]);
[ (Z(2)^0)*a*b*a^5+(Z(2)^0)*b^-1*a^3+(Z(2)^0)*a^3+(Z(2)^0)*b^-1*a^6+(Z(2)^
  0)*a^6+(Z(2)^0)*a*b*a^8, (Z(2)^0)*b*a^9+(Z(2)^0)*a^-1*b^-1*a^4+(Z(2)^0)*a^
  -1*b^-1*a^7+(Z(2)^0)*a^3+(Z(2)^0)*a^6+(Z(2)^0)*b*a^6,
  (Z(2)^0)*a^2*b*a^4+(Z(2)^0)*a*b^-1*a^5+(Z(2)^0)*a^2*b*a^7+(Z(2)^0)*a^3+(
  Z(2)^0)*a*b^-1*a^2+(Z(2)^0)*a^6 ]
```

Using the first primitive idempotent  $e$ , we can consider the left ideal  $\mathbb{F}_2 Ge$  of  $\mathbb{F}_2 G$  and compute its corresponding code.

```
gap> e:=P[1];;
gap> CodeWordByGroupRingElement(F, S, e);
```

```

[ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0,
  Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0, Z(2)^0, 0*Z(2),
  0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2) ]
gap> FGe := LeftIdealByGenerators(FG,[e]);
gap> V := VectorSpace(F,CodeByLeftIdeal(F,G,S,FGe));
gap> B := Basis(V);
gap> code := GeneratorMatCode(B,F);
a linear [27,6,1..6]8..13 code defined by generator matrix over GF(2)
gap> MinimumDistance(code);
6

```

In this way we have constructed a linear  $[27, 6, 6]$ -code over  $\mathbb{F}_2$  by means of a minimal left nilpotent metacyclic group code.

The following is an example of a metacyclic group code which turns out to be a best linear code (as one can check using [BBC<sup>+</sup>12], [Bro98] or [Gra07]), i.e. a code which reaches the maximum bound on the minimum distance. This allows an alternative construction of a linear  $[21, 3, 12]$ -code over  $\mathbb{F}_2$ .

**Example V.2.** We consider the group ring  $\mathbb{F}_2G$  over the group  $G = \langle a, b \mid a^7 = 1, b^3 = 1, ba = a^4b \rangle$ .

```

gap> A:=FreeGroup("a","b");a:=A.1;b:=A.2;;
gap> G:=A/[a^7,b^3,b*a*b^(-1)*a^(-4)];
gap> IsNilpotent(G);
false
gap> F:=GF(2);
gap> FG:=GroupRing(F,G);
gap> S:=AsSet(G);
gap> H:=StrongShodaPairs(G)[3][1];
gap> K:=StrongShodaPairs(G)[3][2];
gap> N:=Normalizer(G,K);
gap> epi:=NaturalHomomorphismByNormalSubgroup(N,K);
gap> QHK:=Image(epi,H);
gap> gq:=MinimalGeneratingSet(QHK)[1];
gap> C:=CyclotomicClasses(Size(F),Index(H,K))[2];
[ 1, 2, 4 ]
gap> P:=PrimitiveIdempotentsTrivialTwisting(FG,H,K,C,[epi,gq]);
[ (Z(2)^0)*a^21+(Z(2)^0)*b*a^21+(Z(2)^0)*b^-1*a^21+(Z(2)^0)*a^22+(Z(2)^0)*b^-1*a^12*b^-1*a^8*b*a^8+(Z(2)^0)*b*a^12*b^-1*a^8*b*a^8+(Z(2)^0)*b^-1*a*b^-1*a^21*b*a^8+(Z(2)^0)*b^-1*a^2*b^-1*a^18*b^-1*a^8*b^-1*a^18+(Z(2)^0)*b*a^2*b^-1*a^18*b^-1*a^8*b^-1*a^18+(Z(2)^0)*a^2*b^-1*a^18*b^-1*a^8*b^-1*a^18,
  (Z(2)^0)*b^-1*a^9*b*a^17+(Z(2)^0)*b^-1*a^7*b^-1*a^28+(Z(2)^0)*a^22+(Z(2)^0)*a^5*b^-1*a^30+(Z(2)^0)*b^-1*a^18*b*a^26+(Z(2)^0)*b^-1*a^8*b^-1*a*b^-1*a^18*b^-1*a^8+(Z(2)^0)*b^-1*a^7*b^-1*a^7*b*a^18+(Z(2)^0)*b^-1*a^9*b*a*b*a^15+(Z(2)^0)*b^-1*a^9*b*a^8*b*a^9+(Z(2)^0)*b^-1*a^8*b^-1*a^27,
  (Z(2)^0)*a^26*b^-1*a^4*b*a^8+(Z(2)^0)*b*a^18*b^-1*a^8*b^-1*a^9+(Z(2)^0)*b*a^27*b*a^4*b*a^18+(Z(2)^0)*a^5*b*a^18*b^-1*a^2*b*a^18+(Z(2)^0)*b^-1*a^22*b*a^18+(Z(2)^0)*b*a^27*b*a^22+(Z(2)^0)*b*a^18*b^-1*a^8*b^2*a^18+(Z(2)^0)*a^20*b*a^18+(Z(2)^0)*b*a^33+(Z(2)^0)*b*a^18*b^-1*a^8*b^2*a^7+(Z(2)^0)*a^38+(Z(2)^0)*b*a^8*b^-1*a^18*b*a^15 ]
gap> e:=P[1];
gap> FGe := LeftIdealByGenerators(FG,[e]);
gap> V := VectorSpace(F,CodeByLeftIdeal(F,G,S,FGe));
gap> B := Basis(V);
gap> code := GeneratorMatCode(B,F);
a linear [21,3,1..12]9..10 code defined by generator matrix over GF(2)
gap> MinimumDistance(code);
12
gap> LowerBoundMinimumDistance(21,3,2);

```

```

12
gap> UpperBoundMinimumDistance(21,3,2);
12

```

In this way we have constructed a best linear  $[21, 3, 12]$ -code over  $\mathbb{F}_2$  by means of a minimal left metacyclic group code.

The following is an example of an abelian-by-metacyclic group code. This allows an alternative construction of a linear  $[105, 3, 60]$ -code over  $\mathbb{F}_2$ .

**Example V.3.** We consider the group ring  $\mathbb{F}_2G$  over the group

$$G = \langle a, b \mid a^7 = 1, b^3 = 1, c^5 = 1, ba = a^4b, [a, c] = 1, [b, c] = 1 \rangle.$$

```

gap> A:=FreeGroup("a","b","c");a:=A.1;b:=A.2;c:=A.3;;
gap> G:=A/[a^7,b^3,c^5,b*a*b^(-1)*a^(-4),c*a*c^(-1)*a^(-1),c*b*c^(-1)*b^(-1)];;
gap> F:=GF(2);;
gap> FG:=GroupRing(F,G);;
gap> S:=AsSet(G);;
gap> H:=StrongShodaPairs(G)[5][1];;
gap> K:=StrongShodaPairs(G)[5][2];;
gap> N:=Normalizer(G,K);;
gap> epi:=NaturalHomomorphismByNormalSubgroup(N,K);;
gap> QHK:=Image(epi,H);;
gap> gq:=MinimalGeneratingSet(QHK)[1];;
gap> C:=CyclotomicClasses(Size(F),Index(H,K))[2];
[ 1, 2, 4 ]
gap> P:=PrimitiveIdempotentsTrivialTwisting(FG,H,K,C,[epi,gq]);;
gap> e:=P[1];;
gap> FGe := LeftIdealByGenerators(FG,[e]);;
gap> V := VectorSpace(F,CodeByLeftIdeal(F,G,S,FGe));;
gap> B := Basis(V);;
gap> code := GeneratorMatCode(B,F);
a linear [105,3,1..60]51..52 code defined by generator matrix over GF(2)
gap> MinimumDistance(code);
60
gap> LowerBoundMinimumDistance(105,3,2);
60
gap> UpperBoundMinimumDistance(105,3,2);
60

```

In this way we have constructed a best  $[105, 3, 60]$ -code by means of a minimal left abelian-by-metacyclic group code.

The following example is one of a metacyclic group code. This allows an alternative construction of a linear  $[20, 4, 12]$ -code over  $\mathbb{F}_3$ .

**Example V.4.** We consider the group ring  $\mathbb{F}_3G$  over the group

$$G = \langle a, b \mid a^5 = 1, b^4 = 1, ba = a^2b \rangle.$$

```

gap> A:=FreeGroup("a","b");a:=A.1;b:=A.2;;
gap> G:=A/[a^5,b^4,b*a*b^(-1)*a^(-2)];;
gap> F:=GF(3);;
gap> FG:=GroupRing(F,G);;
gap> S:=AsSet(G);;
gap> H:=StrongShodaPairs(G)[4][1];;
gap> K:=StrongShodaPairs(G)[4][2];;

```

```

gap> N:=Normalizer(G,K);;
gap> epi:=NaturalHomomorphismByNormalSubgroup(N,K);;
gap> QHK:=Image(epi,H);;
gap> gq:=MinimalGeneratingSet(QHK)[1];;
gap> C:=CyclotomicClasses(Size(F),Index(H,K))[2];
[ 1, 3, 4, 2 ]
gap> P:=PrimitiveIdempotentsTrivialTwisting(FG,H,K,C,[epi,gq]);;
gap> e1:=P[1];;
gap> FGe1 := LeftIdealByGenerators(FG,[e1]);;
gap> V1 := VectorSpace(F,CodeByLeftIdeal(F,G,S,FGe1));;
gap> B1 := Basis(V1);;
gap> code1 := GeneratorMatCode(B1,F);
a linear [20,4,1..8]8..13 code defined by generator matrix over GF(3)
gap> MinimumDistance(code1);
8
gap> e2:=P[2];;
gap> FGe2 := LeftIdealByGenerators(FG,[e2]);;
gap> V2 := VectorSpace(F,CodeByLeftIdeal(F,G,S,FGe2));;
gap> B2 := Basis(V2);;
gap> code2 := GeneratorMatCode(B2,F);
a linear [20,4,1..12]8..13 code defined by generator matrix over GF(3)
gap> MinimumDistance(code2);
12
gap> LowerBoundMinimumDistance(20,4,3);
12
gap> UpperBoundMinimumDistance(20,4,3);
12

```

In this way we have constructed a best  $[20, 4, 12]$ -code by means of a minimal left metacyclic group code. Notice that the choice of the primitive idempotent is crucial to obtain a best code.

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